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A DECOMPOSITION THEOREM FOR THE SPECTRAL SEQUENCE OF LIE FOLIATIONS

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ABSTRACT. For a Lie $\mathfrak{g}$-foliation $\mathcal{F}$ on a closed manifold M, there is an "infinitesimal action of $\mathfrak{g}$ on M up to homotopy along the leaves", in general it is not an action but defines an action of the corresponding connected simply connected Lie group $\mathfrak{G}$ on the term E_1 of the spectral sequence associated to $\mathcal{F}$. Even though E_1 in general is infinite-dimensional and non-Hausdorff (with the topology induced by the $\mathcal{C}^\infty$-topology), it is proved that this action can be averaged when $\mathfrak{G}$ is compact, obtaining a tensor decomposition theorem of E_2. It implies duality in the whole term E_2 for Riemannian foliations on closed oriented manifolds with compact semisimple structural Lie algebra.

INTRODUCTION

The spectral sequence (E_i, d_i) associated to a smooth foliation $\mathcal{F}$ on a manifold M is defined for example in [Sa, KT2, and Se2]. The $\mathcal{C}^\infty$-topology in the space of differential forms induces in E_1 a topology which in general is not Hausdorff [Ha1], obtaining two new bigraded differential algebras: The closure $\overline{O_{E_1}}$ of the trivial subspace in E_1 and the quotient $\mathcal{E}_1 = E_1/\overline{O_{E_1}}$ [He and A2].

If $\mathcal{F}$ is Riemannian and M is compact there are several papers studying the finite-dimensional character and the duality in E_2, $\mathcal{E}_2 = H(\mathcal{E}_1)$ and $H(\overline{O_{E_1}})$, or in parts of them: [Sa, EH, ESH, Se1, Se2, He, KT1–KT3, A1 and A2]. For this type of foliations, with M oriented, it is proved in [A2] that the de Rham duality map induces in $H(\overline{O_{E_1}})$ and $\mathcal{E}_2$ different types of duality, so the possibility of obtaining duality in E_2 depends on the properties of the canonical long exact sequence which relates these three homologies. For example, in [A2] it is proved that $E_2 \cong \mathcal{E}_2$ when the leaves are compact.

In the present paper this last result is generalized (§4), obtaining that $E_2 \cong \mathcal{E}_2$ when the structural Lie algebra of $\mathcal{F}$ is compact and semisimple. Using the structure theorems of P. Molino for Riemannian foliations [Mo] its proof is reduced to the case of Lie foliations, where it is a consequence of the main result of this paper (§3): *For Lie $\mathfrak{g}$-foliations on manifolds with $\mathfrak{g}$ compact and semisimple we have $E_2 \cong E_2^{0,\cdot} \otimes H^\cdot(\mathfrak{g})$*. The analogous decomposition for $\mathcal{E}_2$ is proved in [A2] with easier arguments.

Let $\mathcal{F}$ be a Lie $\mathfrak{g}$-foliation on a compact manifold M, and let $\mathfrak{G}$ be the connected simply connected Lie group with Lie algebra $\mathfrak{g}$. By choosing a com-

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plement $\nu \subset TM$ of the subbundle of vectors tangent to $\mathscr{F}$ we can define a smooth map $\phi: M \times \mathfrak{g} \to M$ (§1), which induces an infinitesimal action only if ν is completely integrable. So ϕ does not induce an action of $\mathfrak{G}$ neither on M nor on its de Rham complex $A(M)$, but it induces an action on E_1. The proof of the above tensor decomposition of E_1 follows directly whenever the above action can be "averaged". If $\mathfrak{g}$ is compact and semisimple then $\mathfrak{G}$ is compact, but even in this case an averaging process cannot be easily made because, in general, E_1 is an infinite-dimensional non-Hausdorff topological vector space, (the integration of continuous functions of $\mathfrak{G}$ to E_1 is not well defined).

The dual action of $\mathfrak{G}$ on E_1' can be averaged [B], but in this case we obtain the tensor decomposition of $H(E_1') \equiv \mathscr{E}_2' \cong \mathscr{E}_2$. Thus it would be another proof of Theorem (9.10) of [A2].

For the "basic complex", $E_1^{\bullet,0}$, and the "transverse complex", $E_1^{\bullet,p}$ ($p = \dim(\mathscr{F})$), the situation is much simpler, for they can be identified with a subspace of $A(\mathfrak{G})$ and a quotient of $A_c(\mathfrak{G})$ respectively, where the above action of $\mathfrak{G}$ is given by the left translations [Ha1 and He]. So, the averaging process on $E_1^{\bullet,0}$ and $E_1^{\bullet,p}$ can be easily made when $\mathfrak{g}$ is compact or nilpotent, obtaining $E_2^{\bullet,0} \cong E_2^{\bullet,p} \cong H^{\bullet}(\mathfrak{g})$.

The main part of this paper is devoted to obtain an averaging process of the above action on the whole E_1 when $\mathfrak{g}$ is compact and semisimple. To do that, some parts of measure zero are modified in the closure of the domain of injectivity of the exponential map to define new compact spaces (§2), so that ϕ can be "lift" to them, where the failure of being "like an action" can be controlled (§3). Then, by integrating maps induced by the lifting of ϕ on those compact spaces, we obtain an operator on $A(M)$ which defines an "averaging retraction" of E_1 onto its subspace of elements that are invariant by the action of $\mathfrak{G}$. This retraction induces an isomorphism in cohomology from which the above decomposition theorem follows.

From the arguments of this proof, when $\mathscr{F}$ is orientable, we obtain an explicit expression of the volume form along the leaves induced by an orientation and the restriction to the leaves of a Riemannian metric on M for which all the leaves of $\mathscr{F}$ are minimal submanifolds.

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1. Preliminaries

Let M be a smooth manifold which carries a smooth foliation $\mathscr{F}$ of dimension p and codimension q, and let $T\mathscr{F} \subset TM$ be the subbundle of vectors of M tangent to $\mathscr{F}$.

The de Rham differential algebra $(A(M), d)$ (or simply (A, d)) of M is filtered by differential ideals so that a differential form of degree r is said to be of filtration $\geq k$ if it vanishes whenever $r - k + 1$ of the vectors are tangent to $\mathscr{F}$. In this way we obtain the spectral sequence $(E_i(\mathscr{F}), d_i)$ (or simply (E_i, d_i)) which converges after a finite number of steps to the de Rham cohomology of M.

For each subbundle $\nu \subset TM$, complementary of $T\mathscr{F}$, we bigrade A by setting

$$A^{u,v} = \Gamma(\Lambda^v T^*\mathscr{F} \otimes \Lambda^u \nu^*)$$

for u, v integers, obtaining the decomposition of d as sum of the bihomogeneous operators $d_{0,1}$, $d_{1,0}$ and $d_{2,-1}$, where the double subindices denote the corresponding bidegrees, verifying

$$\begin{gathered} d_{0,1}^2 = d_{2,-1}^2 = d_{0,1}\circ d_{1,0} + d_{1,0}\circ d_{0,1} = 0, \\ d_{1,0}\circ d_{2,-1} + d_{2,-1}\circ d_{1,0} = d_{1,0}^2 + d_{2,-1}\circ d_{0,1} + d_{0,1}\circ d_{2,-1} = 0. \end{gathered} \tag{1.1}$$

And we have the following canonical identities of bigraded differential algebras

$$(E_0, d_0) \equiv (A, d_{0,1}), \quad (E_1, d_1) \equiv (H(A, d_{0,1}), d_{1,0*}). \tag{1.2}$$

(A, d) is a topological differential algebra with the $\mathscr{C}^\infty$-topology, then each (E_i, d_i) is another topological differential algebra with the induced topology and the identities (1.2) are also topological.

E_1 in general is not Hausdorff [Ha1] obtaining the bigraded differential algebras $\overline{O_{E_1}}$ (the closure in E_1 of the trivial subspace) and $\mathscr{E}_1 = E_1/\overline{O_{E_1}}$. Let $\mathscr{E}_2 = H(\mathscr{E}_1)$.

If $\mathscr{F}$ is Riemannian and M compact then E_2, $\mathscr{E}_2$, and $H(\overline{O_{E_1}})$ are of finite dimension [Se2, A1 and A2], and if M is also oriented we have the duality isomorphisms [A2]

$$\mathscr{E}_2^{u,v} \cong \mathscr{E}_2^{q-u,q-v}, \quad H^{u,v}(\overline{O_{E_1}}) \cong H^{q-u-1,p-v+1}(\overline{O_{E_1}}) \tag{1.3}$$

for u, v integers, obtaining the canonical isomorphism

$$E_2^{q,\cdot} \cong \mathscr{E}_2^{q,\cdot}. \tag{1.4}$$

Suppose that $\mathscr{F}$ is a Lie $\mathfrak{g}$-foliation and M is compact. If $\mathfrak{G}$ is the simply connected Lie group with Lie algebra $\mathfrak{g}$ then [Mo] there exists a covering map $\pi: \widetilde{M} \to M$ and a fibre bundle $D: \widetilde{M} \to \mathfrak{G}$ such that:

(i) The leaves of $\pi^*\mathscr{F}$ are the fibres of D.
(ii) There exists an injective homomorphism $h: \operatorname{Aut}(\pi) \to \mathfrak{G}$ so that D is h-equivariant ($D\circ\zeta(\tilde{x}) = h(\zeta)\cdot D(\tilde{x})$ for $\tilde{x}\in\widetilde{M}$ and $\zeta \in \operatorname{Aut}(\pi)$).

Choose $\nu \subset TM$ as above and the corresponding bigradation in A. Let $\tilde{\nu} = \pi^*\nu \subset T\widetilde{M}$ and $\widetilde{\mathscr{F}} = \pi^*\mathscr{F}$. For each field $X \in \mathfrak{g}$ there exists a unique field $\widetilde{X}^\nu \in \Gamma\tilde{\nu}$ such that $D_*\circ\widetilde{X}^\nu = X\circ D$. Since D is h-equivariant $\widetilde{X}^\nu$ is $\operatorname{Aut}(\pi)$-invariant, so it determines a field $X^\nu \in \Gamma\nu \subset \mathfrak{X}(M)$ which is an infinitesimal transformation of $\mathscr{F}$. Let θ_X and i_X be the Lie derivative $\mathscr{L}_{X^\nu}$ and the interior product i_{X^ν} respectively, obtaining

$$\begin{gathered} d_{0,1}\circ i_X + i_X\circ d_{0,1} = 0, \quad (\theta_X)_{0,0}\circ d_{0,1} = d_{0,1}\circ(\theta_X)_{0,0}, \\ i_{[X,Y]} = (\theta_X)_{0,0}\circ i_Y - i_Y\circ(\theta_X)_{0,0}, \quad (\theta_X)_{0,0} = d_{1,0}\circ i_X + i_X\circ d_{1,0}, \\ (\theta_{[X,Y]})_{0,0} = (\theta_X\circ\theta_Y - \theta_Y\circ\theta_X)_{0,0} - d_{0,1}\circ i_{\Omega(X\wedge Y)} - i_{\Omega(X\wedge Y)}\circ d_{0,1}, \end{gathered} \tag{1.5}$$

where $\Omega: \Lambda^2\mathfrak{g} \to \Gamma T\mathscr{F}$ is the linear map defined by $\Omega(X\wedge Y) = [X^\nu, Y^\nu] - [X, Y]^\nu$. Therefore we get the operation $(\mathfrak{g}, i_1, \theta_1, E_1, d_1)$, where $i_{1_X} \equiv i_{X*}$ and $\theta_{1_X} \equiv (\theta_X)_{0,0*}$ by (1.2), with the algebraic connection $\chi: \mathfrak{g}^* \to E_1^{1,0} \subset A^{1,0}$ where $\chi(\alpha)$ is determined by $\pi^*\chi(\alpha) = D^*\alpha$ for each $\alpha\in\mathfrak{g}^*\subset A^1(\mathfrak{G})$. Then it follows that [Ma and A2]

$$E_2^{u,v} \cong H^u(\mathfrak{g}; \theta_1: \mathfrak{g} \to \operatorname{End}(E_1^{0,v})) \quad \text{for } u, v \text{ integers}. \tag{1.6}$$

The above operation and algebraic connection induces an operation of $\mathfrak{g}$ on $\mathscr{E}_1$ with the corresponding algebraic connection and analogous consequences.

For each (complete) vector field X we denote by X_t $(t \in \mathbb{R})$ the corresponding one-parameter group of transformations. Then we have the smooth maps $\phi: M \times \mathfrak{g} \to M$ and $\tilde{\phi}: \widetilde{M} \times \mathfrak{g} \to \widetilde{M}$, depending on $\tilde{\nu}$, defined by $\phi(x, X) = X_1^\nu(x)$ and $\tilde{\phi}(\tilde{x}, X) = \tilde{X}_1^\nu(\tilde{x})$, obtaining the commutativity of the diagrams

$$(1.7) \qquad \begin{array}{ccc} \widetilde{M} \times \mathfrak{g} & \xrightarrow{\tilde{\phi}} & \widetilde{M} \\ {\scriptstyle \pi\times \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \pi} \\ M \times \mathfrak{g} & \xrightarrow{\phi} & M \end{array} \qquad \begin{array}{ccc} M \times \mathfrak{g} & \xrightarrow{\tilde{\phi}} & M \\ {\scriptstyle D\times \exp}\downarrow & & \downarrow{\scriptstyle D} \\ \mathfrak{G} \times \mathfrak{G} & \xrightarrow{\mu} & \mathfrak{G} \end{array}$$

where μ is the operation of $\mathfrak{G}$. Let $\phi_X = \phi(\cdot, X): M \to M$ for each $X \in \mathfrak{g}$.

2. Compact spaces associated to a compact Lie group

Let N be a connected complete Riemannian manifold. For each $x \in N$ let $C(x) \subset N$ and $C^*(x) \subset T_xN$ be the corresponding cut locus and tangential cut locus respectively [Ko and Kl], let $B^*(x)$ be the radial domain in T_xN bounded by $C^*(x)$ and let $B(x) = \exp_x(B^*(x))$.

Proposition 2.1 [Ko and Kl]. (i) $C(x) = \partial B(x) = N - B(x)$.

(ii) $\exp_x: B^*(x) \to B(x)$ *is a diffeomorphism.*

(iii) $C(x)$ *and* $C^*(x)$ *have Lebesgue measure zero.*

(iv) *If* N *is compact then so is* $\overline{B^*(x)}$.

Let $\mathfrak{G}$ be a connected compact Lie group with Lie algebra $\mathfrak{g}$ and consider the canonical biinvariant metric on $\mathfrak{G}$ (Chapter 6 of [P]). For the identity element $e \in \mathfrak{G}$ let $B^* = B^*(e) \subset \mathfrak{g}$, $B = B(e) \subset \mathfrak{G}$ and $\log = \exp^{-1}: B \to B^*$. Then we have the compact space

$$F = \{(X, Y, Z) \in \overline{B^{*3}} / \exp(X)\cdot\exp(Y) = \exp(Z)\} \subset \mathfrak{g}^3,$$

and for each $X \in \overline{B^*}$ we also have the compact space

$$F_X = \{(Y, Z) \in \mathfrak{g}^2 / (X, Y, Z) \in F\} \subset \mathfrak{g}^2.$$

We can speak of smoothness of geometrical objects on F (resp. F_X) as subspace of $\mathfrak{g}^3$ (resp. $\mathfrak{g}^2$).

Let $i: \mathfrak{g}^2 \to \mathfrak{g}^2$ be the involution $(Y, Z) \mapsto (Z, Y)$ and let $\mathrm{pr}_1: \mathfrak{g}^2 \to \mathfrak{g}$ be the canonical first projection. Denoting $a = \exp(X)$ we have the smooth map

$$\begin{aligned} \mathcal{I}_X: B \cap L_a^{-1}B &\to F_X \\ g &\mapsto (\log(g), \log(a\cdot g)) \end{aligned}$$

and let W_X be its image.

Proposition 2.2. (i) W_X *is open in* F_X *and* $\mathcal{I}_X: B \cap L_a^{-1}B \to W_X$ *is a diffeomorphism.*

(ii) *In general* $\overline{W_X} \neq F_X$. $\overline{W_X}$ *is formed by the pairs* $(Y, Z) \in F_X$ *such that*

$$(2.1) \qquad Z \in \overline{B^* \cap \exp^{-1} L_a^{-1} \exp(U \cap B^*)}$$

for all neighborhoods U *of* Y *in* $\mathfrak{g}$.

(iii) $i(F_X) = F_{-X}$ *and we have the commutative diagram*

$$\begin{array}{ccc} B \cap L_a^{-1}B & \xrightarrow{\mathcal{I}_X} & F_X \\ {\scriptstyle L_a}\downarrow & & \downarrow{\scriptstyle i} \\ B \cap L_a B & \xrightarrow{\mathcal{I}_{-X}} & F_{-X} \end{array} \tag{2.2}$$

Proof. (i) We have $W_X = F_X \cap (\log(B \cap L_a^{-1}B) \times \mathfrak{g})$ which is open in F_X, and it is easy to check that $\exp \circ \mathrm{pr}_1 \colon W_X \to B \cap L_a^{-1}B$ is the inverse map of $\mathcal{I}_X \colon B \cap L_a^{-1}B \to W_X$.

(ii) If Y and Z are different points in $\overline{B^*}$ such that $\exp(Y) = \exp(Z)$ then $(Y, Z) \in F_0 - \overline{W_0}$.

Let $(Y, Z) \in \overline{W_X}$ and let U be a neighborhood of Z in $\mathfrak{g}$. Take a sequence $((Y_n, Z_n))_{n\in\mathbb{N}}$ in W_X converging to (Y, Z). We can assume that $Y_n \in U$ for all $n \in \mathbb{N}$ obtaining that $Z_n \in B^* \cap \exp^{-1} L_a \exp(U \cap B^*)$, so (2.1) is verified.

Let $(Y, Z) \in F_X$ such that (2.1) is verified for all neighborhood U of Y. If $\{U_n\}_{n\in\mathbb{N}}$ and $\{V_n\}_{n\in\mathbb{N}}$ are neighborhood bases of Y and Z respectively then for each $n \in \mathbb{N}$ there exists a point $Z_n \in V_n \cap B^* \cap \exp^{-1} L_a \exp(U_n \cap B^*)$, obtaining that $g_n = a^{-1} \cdot \exp(Z_n) \in B \cap L_a^{-1}B$ and $(\mathcal{I}_X(g_n))_{n\in\mathbb{N}}$ converges to (Y, Z), so $(Y, Z) \in \overline{W_X}$.

(iii) Since $\exp(-X) = a^{-1}$ we have $i(F_X) = F_{-X}$, and the commutativity of (2.2) has a straightforward verification. □

For X, $Y \in \overline{B^*}$ we have the open subset $W_{X,Y} = \mathcal{I}_X(B \cap L_a^{-1}B \cap L_b^{-1}B) \subset F_X$, where $a = \exp(X)$ and $b = \exp(Y)$, obtaining the diffeomorphism $\mathcal{I}_{X,Y} = \mathcal{I}_Y \circ \mathcal{I}_X^{-1} \colon W_{X,Y} \to W_{Y,X}$.

Assume $\mathfrak{G}$ oriented and let Δ be the unique biinvariant volume form on $\mathfrak{G}$ such that $\int_{\mathfrak{G}} \Delta = 1$, which induces a Haar measure μ on $\mathfrak{G}$. Then, for each $X \in \overline{B^*}$ let μ_X be the Borel measure on F_X, concentrated on W_X, where it corresponds to μ by $\mathcal{I}_X$.

Proposition 2.3. *We have*:

(i) $\mu_X(F_X) = \mu_X(W_X) = \mu_X(W_{X,Y}) = \mu(B \cap L_a^{-1}B \cap L_b^{-1}B) = \mu(B \cap L_a^{-1}B) = \mu(\mathfrak{G}) = 1$.

(ii) μ_X *corresponds to* μ_{-X} *by* $i \colon F_X \to F_{-X}$.

(iii) μ_X *corresponds to* μ_Y *by* $\mathcal{I}_{X,Y} \colon W_{X,Y} \to W_{Y,X}$.

Proof. (i) follows from Proposition 2.1, (ii) follows from the commutativity of (2.2), and (iii) is true because $\mathcal{I}_{X,Y} = \mathcal{I}_Y \circ \mathcal{I}_X^{-1}$. □

3. Decomposition of E_2

Let $\mathcal{F}$ be a Lie $\mathfrak{g}$-foliation with $\mathfrak{g}$ compact and semisimple, then the simply connected Lie group $\mathfrak{G}$ with Lie algebra $\mathfrak{g}$ is compact (6.22 of [P]). Examples of such a $\mathfrak{g}$ are $\mathfrak{su}(n)$ and $\mathfrak{sp}(n)$ for all $n \in \mathbb{N}$, and $\mathfrak{so}(n)$ for $n > 2$ (6.28, 6.47 and 6.48 of [P]). The notation of the §§1 and 2 remains here. Then, for a fixed subbundle $\nu \subset TM$, complementary of $T\mathcal{F}$, we define the operators ρ and λ on A by setting

$$\rho(\alpha) = \int_{\overline{B^*}} \phi_X^* \alpha \cdot \Delta^*(X), \quad \lambda(\alpha) = \int_{\overline{B^*}} \Phi_X \alpha \cdot \Delta^*(X),$$

where $\Delta^* = \exp^*\Delta$ and Φ_X is the homogeneous operator of degree -1 on A associated to the homotopy ϕ_{tX} $(t \in I = [0, 1])$. ρ and λ are linear homogeneous of degrees 0 and -1 respectively, being $\rho - \mathrm{id} = d \circ \lambda + \lambda \circ d$. Moreover, since ϕ_{tX} preserves the foliation (because X^ν is an infinitesimal transformation of $\mathscr{F}$) Φ_X reduces the filtration at most by a unity, so ρ is filtration-preserving and λ reduces the filtration at most by a unity. Therefore they induce bihomogeneous linear operators ρ_1 and λ_1 on E_1 of bidegrees $(0, 0)$ and $(-1, 0)$ respectively such that $\rho_1 - \mathrm{id} = \lambda_1 \circ d_1 + d_1 \circ \lambda_1$ $(\rho_1 \equiv \rho_{0,0*}$ and $\lambda_1 \equiv \lambda_{-1,0*}$ by (1.2)).

If F is the compact space associated to $\mathfrak{G}$ (§2) we define the continuous maps σ, $\eta\colon F \times I \to \mathfrak{G}$ by setting

$$\sigma(\xi, t) = \exp(tZ),$$
$$\eta(\xi, t) = \begin{cases} \exp(2tX) & \text{if } t \in I_1 = [0, 1/2], \\ \exp(X)\cdot\exp((2t-1)Y) & \text{if } t \in I_2 = [1/2, 1], \end{cases}$$

where $\xi = (X, Y, Z) \in F$. σ is smooth and so are the restrictions of η to each $F \times I_i$ $(i = 1, 2)$.

For each $\xi_0 = (X_0, Y_0, Z_0) \in F$ we have the paths $\sigma(\xi_0, \cdot)$, $\eta(\xi_0, \cdot)\colon I \to \mathfrak{G}$ of the identity element $e \in \mathfrak{G}$ to $\exp(Z_0)$, the first one is smooth and the second one has smooth restrictions to each I_i. So there exists a homotopy $H_{\xi_0}\colon I \times I \to \mathfrak{G}$ of $\sigma(\xi_0, \cdot)$ to $\eta(\xi_0, \cdot)$, relative to $\{0, 1\}$, with smooth restrictions to each $I_i \times I$.

We define the maps σ_{ξ_0}, $\eta_{\xi_0}\colon F \times I \to \mathfrak{G}$ by setting

$$\sigma_{\xi_0}(\xi, t) = \sigma(\xi_0, t)^{-1}\cdot\sigma(\xi, t), \qquad \eta_{\xi_0}(\xi, t) = \eta(\xi_0, t)^{-1}\cdot\eta(\xi, t),$$

and let U be a contractile neighborhood of e in $\mathfrak{G}$. Thus, since $\sigma_{\xi_0}(\{\xi_0\}\times I) = \eta_{\xi_0}(\{\xi_0\}\times I) = \{e\}$ and I is compact there exists a neighborhood Q_{ξ_0} of ξ_0 in F so that $\sigma_{\xi_0}(Q_{\xi_0}\times I)$, $\eta_{\xi_0}(Q_{\xi_0}\times I) \subset U$. Moreover, we have

$$\text{(3.1)} \qquad \begin{aligned} \sigma_{\xi_0}(\cdot, 0) &= \eta_{\xi_0}(\cdot, 0) = \mathrm{const}_e, \\ \sigma_{\xi_0}(\xi, 1) &= \eta_{\xi_0}(\xi, 1) = \exp(Z_0)^{-1}\cdot\exp(Z), \end{aligned}$$

where $\xi = (X, Y, Z) \in F$. Then, by means of a smooth contraction of U we obtain a continuous map $S_{\xi_0}\colon Q_{\xi_0}\times I \times I \to U$ with smooth restrictions to each $Q_{\xi_0}\times I_i \times I$ so that

$$\text{(3.2)} \qquad \begin{aligned} &S_{\xi_0}(\cdot, \cdot, 0) = \sigma_{\xi_0}, \quad S_{\xi_0}(\cdot, \cdot, 1) = \eta_{\xi_0}, \\ &S_{\xi_0}(\cdot, 0, \cdot) = \mathrm{const}_e, \quad S_{\xi_0}(\xi, 1, s) = \exp(Z_0)^{-1}\cdot\exp(Z), \end{aligned}$$

for $\xi = (X, Y, Z) \in Q_{\xi_0}$.

Let $\overline{H}_{\xi_0}\colon Q_{\xi_0}\times I \times I \to \mathfrak{G}$ be the map defined by

$$\overline{H}_{\xi_0}(\xi, t, s) = H_{\xi_0}(t, s)\cdot S_{\xi_0}(\xi, t, s).$$

Then it is easy to check that

$$\text{(3.3)} \qquad \begin{aligned} &\overline{H}_{\xi_0}(\cdot, \cdot, 0) = \sigma_{|\sigma_{\xi_0}\times I}, \quad \overline{H}_{\xi_0}(\cdot, \cdot, 1) = \eta_{|Q_{\xi_0}\times I}, \\ &\overline{H}_{\xi_0}(\cdot, 0, \cdot) = \mathrm{const}_e, \quad \overline{H}_{\xi_0}(\xi, 1, s) = \exp(Z), \end{aligned}$$

for $\xi = (X, Y, Z) \in Q_{\xi_0}$.

Since F is compact there exist a finite number of points $\xi_j = (X_j, Y_j, Z_j) \in F$ $(j = 1, \dots, k)$ such that $F = Q_{\xi_1}\cup\cdots\cup Q_{\xi_k}$. Let $Q_j = Q_{\xi_j}$ and $\overline{H}_j = \overline{H}_{\xi_j}$.

Lemma 3.1. *For each j there exists a unique continuous map $\widetilde{H}_j\colon \widetilde{M}\times Q_j\times I\times I\to\widetilde{M}$ with smooth restrictions to each $\widetilde{M}\times Q_j\times I_i\times I$ such that*

(i) $D\circ\widetilde{H}_j(\tilde{x},\xi,t,s)=D(\tilde{x})\cdot\overline{H}_j(\xi,t,s)$,

(ii) $\widetilde{H}_j(\tilde{x},\xi,0,s)=\tilde{x}$,

(iii) $(d/dt)\widetilde{H}_j(\tilde{x},\xi,t,s)\in\tilde{\nu}$ *for* $t\neq 1/2$.

Proof. For $i=1,2$ and $j=1,\dots,k$ we consider the smooth maps

$$J_{i,j}\colon \mathfrak{G}\times Q_j\times I_i\times I\to G$$
$$(g,\xi,t,s)\mapsto g\cdot\overline{H}_j(\xi,t,s).$$

We have the pull-back

$$J_{i,j}^{*}\widetilde{M}=\{(g,\xi,t,s,\tilde{x})\in\mathfrak{G}\times Q_j\times I_i\times I\times\widetilde{M}/D(\tilde{x})=g\cdot\overline{H}_j(\xi,t,s)\}$$

of the fibre bundle $D\colon\widetilde{M}\to\mathfrak{G}$ with the projection $\overline{D}_{i,j}\colon J_{i,j}^{*}\widetilde{M}\to\mathfrak{G}\times Q_j\times I_i\times I$ given by $\overline{D}_{i,j}(g,\xi,t,s,\tilde{x})=(g,\xi,t,s)$ and the homomorphism of fibre bundles $\overline{J}_{i,j}\colon J_{i,j}^{*}\widetilde{M}\to\widetilde{M}$ given by $\overline{J}_{i,j}(g,\xi,t,s,\tilde{x})=\tilde{x}$.

It is easy to prove that for each $(g,\xi,t,s,\tilde{x})\in J_{i,j}^{*}\widetilde{M}$ the homomorphism $\overline{J}_{i,j*}\colon T_{(g,\xi,t,s,\tilde{x})}(J_{i,j}^{*}\widetilde{M})\to T_{\tilde{x}}(\widetilde{M})$ is surjective, then $(\overline{J}_{i,j*})^{-1}\tilde{\nu}\subset T(J_{i,j}^{*}\widetilde{M})$ is a vectorial subbundle, complementary of the vertical subbundle $\operatorname{Ker}(\overline{D}_{i,j*})\subset T(J_{i,j}^{*}\widetilde{M})$. It follows that for the vector field $X_{i,j}=(0,0,\partial/\partial t,0)$ on $\mathfrak{G}\times Q_j\times I_i\times I$ there exists a unique vector field $\widetilde{X}_{i,j}\in\Gamma((\overline{J}_{i,j*})^{-1}\tilde{\nu})$ on $J_{i,j}^{*}\widetilde{M}$ so that $\overline{D}_{i,j*}\circ\widetilde{X}_{i,j}=X_{i,j}\circ\overline{D}_{i,j}$.

For $(\tilde{x},\xi,t,s)\in\widetilde{M}\times Q_j\times I\times I$ we have that $(D(\tilde{x}),\xi,0,s,\tilde{x})\in J_{1,j}^{*}\widetilde{M}$ by (3.3), then we define $\widetilde{H}_j(\tilde{x},\xi,t,s)=\overline{J}_{1,j}\circ(\widetilde{X}_{1,j})_t(D(\tilde{x}),\xi,0,s,\tilde{x})$ if $t\in I_1$. Let $\tilde{x}_1=\overline{J}_{1,j}\circ(\widetilde{X}_{1,j})_{1/2}(D(\tilde{x}),\xi,0,s,\tilde{x})\in\widetilde{M}$, then $(D(\tilde{x}),\xi,1/2,s,\tilde{x}_1)\in J_{2,j}^{*}\widetilde{M}$ and we define $\widetilde{H}_j(\tilde{x},\xi,t,s)=J_{2,j}\circ(\widetilde{X}_{2,j})_{t-1/2}(D(\tilde{x}),\xi,1/2,s,\tilde{x}_1)$ if $t\in I_2$. In this way $\widetilde{H}_j$ is continuous with smooth restrictions to each $\widetilde{M}\times Q_j\times I_i\times I$ and it is easy to check that (i)–(iii) is verified.

Now we prove the unicity of $\widetilde{H}_j$. Take $\tilde{x}\in\widetilde{M}$, $\xi\in Q_j$ and $s\in I$, and let $\gamma\colon I\to\widetilde{M}$ be a curve with smooth restrictions to each I_i such that:

(i′) $D\circ\gamma(t)=D(\tilde{x})\cdot\overline{H}_j(\xi,t,s)$,

(ii′) $\gamma(0)=\tilde{x}$,

(iii′) $(d/dt)\gamma(t)\in\tilde{\nu}$ for $t\neq 1/2$.

For $i=1,2$ we define the smooth maps $\overline{\gamma}_i\colon I_i\to J_{i,j}^{*}\widetilde{M}$ by setting $\overline{\gamma}_i(t)=(D(\tilde{x}),\xi,t,s,\gamma(t))$ (it is well defined by (i′)). We have $\overline{J}_{i,j}\circ\overline{\gamma}_i=\gamma_{|I_i}$, so $(d/dt)\overline{\gamma}_i(t)\in(\overline{J}_{i,j*})^{-1}\tilde{\nu}$ for all $t\in I_i$ (by (iii′)). Moreover, $\overline{D}_{1,j}\circ\overline{\gamma}_1(t)=(X_{1,j})_t(D(\tilde{x}),\xi,0,s)$ for $t\in I_1$ and $\overline{D}_{2,j}\circ\overline{\gamma}_2(t)=(X_{2,j})_{t-1/2}(D(\tilde{x}),\xi,1/2,s)$ for $t\in I_2$. Thus $\overline{\gamma}_1(t)=(\widetilde{X}_{1,j})_t(D(\tilde{x}),\xi,0,s,\tilde{x})$ for $t\in I_1$ and $\overline{\gamma}_2(t)=(\widetilde{X}_{2,j})_{t-1/2}(D(\tilde{x}),\xi,1/2,\gamma(1/2))$ for $t\in I_2$, obtaining $\gamma=\widetilde{H}(\tilde{x},\xi,\cdot,s)$. □

Lemma 3.2. *For each j and each $\xi=(X,Y,Z)\in Q_j$ we have*

(i) $\widetilde{H}_j(\cdot,\xi,1,0)=\tilde{\phi}_Z$,

(ii) $\widetilde{H}_j(\cdot,\xi,1,1)=\tilde{\phi}_Y\circ\tilde{\phi}_X$,

(iii) $\widetilde{H}_j(\tilde{x},\xi,1,s)\in D^{-1}(D(\tilde{x})\cdot\exp(Z))$ *for all* $\tilde{x}\in\widetilde{M}$ *and for all* $s\in I$.

Proof. Fix $\tilde{x} \in \widetilde{M}$ and consider the curves γ_0, $\gamma_1 \colon I \to \widetilde{M}$ given by $\gamma_0(t) = \widetilde{H}_j(\tilde{x}, \xi, t, 0)$ and $\gamma_1(t) = \widetilde{H}_j(\tilde{x}, \xi, t, 1)$. By (i) of Lemma 3.1 and (3.3) we have that $D \circ \gamma_0$, $D \circ \gamma_{1|I_1}$ and $D \circ \gamma_{1|I_2}$ are integral curves of the vector fields Z, $2X$ and $2Y$ respectively. So, by (ii) and (iii) of Lemma 3.1 we have that γ_0, $\gamma_{1|I_1}$ and $\gamma_{1|I_2}$ are integral curves of $\widetilde{Z}^\nu$, $2\widetilde{X}^\nu$ and $2\widetilde{Y}^\nu$ with initial points $\tilde{x}$, $\tilde{x}$ and $\gamma_1(1/2)$ respectively, from which (i) and (ii) follows.

On the other hand, by (i) of Lemma 3.1 and by (3.3) we obtain

$$D \circ \widetilde{H}_j(\tilde{x}, \xi, 1, s) = D(\tilde{x}) \cdot \exp(Z). \quad \square$$

By the unicity of $\widetilde{H}_j$ and by the h-equivariance of D it can be easily proved that

$$\zeta \circ \widetilde{H}_j(\tilde{x}, \xi, t, s) = \widetilde{H}_j(\zeta(\tilde{x}), \xi, t, s) \tag{3.4}$$

for all $\zeta \in \mathrm{Aut}(\pi)$. Thus there exists a smooth map $K_j \colon M \times Q_j \times I \to M$ such that the following diagram is commutative

$$\begin{array}{ccc} \widetilde{M} \times Q_j \times I & \xrightarrow{\widetilde{H}_j(\cdot, \cdot, 1, \cdot)} & \widetilde{M} \\ {\scriptstyle \pi \times \mathrm{id} \times \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \pi} \\ M \times Q_j \times I & \xrightarrow{K_j} & M \end{array} \tag{3.5}$$

Moreover, for all $\xi = (X, Y, Z) \in Q_j$, by Lemma 3.2 and by the commutativity of (1.7) and (3.5) we have that $K_j(\cdot, \xi, \cdot) \colon M \times I \to M$ is an integrable homotopy of ϕ_Z to $\phi_Y \circ \phi_Z$ [E], therefore the corresponding homotopy operator on A preserves the filtration, thus its bihomogeneous component of bidegree $(0, -1)$, $k_{j,\xi} \colon A \to A$, verifies

$$(\phi_X^* \circ \phi_Y^* - \phi_Z^*)_{0,0} = d_{0,1} \circ k_{j,\xi} + k_{j,\xi} \circ d_{0,1}, \tag{3.6}$$

and $k_{j,\xi}(\alpha)$ depends smoothly on $\xi \in Q_j$ for each $\alpha \in A$ fixed.

Take a partition of unity $\{f_j\}_{j=1,\dots,k}$ subordinated to the open covering $\{Q_j\}_{j=1,\dots,k}$ of F. Then the functions $f_j(X, \cdot, \cdot)$ form a partition of unity subordinated to the open covering of F_X given by the subsets $Q_{j,X} = \{(Y, Z) \in \mathfrak{g}^2 / (X, Y, Z) \in Q_j\} \subset F_X$.

For $\alpha \in A$ and $X \in \overline{B^*}$, by Proposition 2.3 we have

$$\phi_X^* \circ \rho(\alpha) = \int_{F_X} \phi_X^* \circ \phi_Y^* \alpha \cdot d\mu_X(Y, X),$$

$$\begin{aligned} \rho(\alpha) &= \int_{W_{X,-X}} \phi_Y^* \alpha \cdot d\mu_X(Y, Z) = \int_{W_{-X,X}} \phi_Y^* \alpha \cdot d\mu_{-X}(Y, Z) \\ &= \int_{F_{-X}} \phi_Y^* \alpha \cdot d\mu_{-X}(Y, Z) = \int_{F_X} \phi_Z^* \alpha \cdot d\mu_X(Y, Z), \end{aligned}$$

thus we get

$$(\phi_X^* \circ \rho - \rho)(\alpha) = \int_{F_X} (\phi_X^* \circ \phi_Y^* - \phi_Z^*)(\alpha) \cdot d\mu_X(Y, Z). \tag{3.7}$$

Therefore, if we define the bihomogeneous linear operator $\Psi_X : A \to A$ by setting

$$\Psi_X(\alpha) = \sum_{j=1}^{k} \int_{Q_{j,X}} k_{j,\xi}(\alpha) \cdot f_j(\xi) \cdot d\mu_X(Y, Z),$$

where $\xi = (X, Y, Z)$ for each $(Y, Z) \in Q_{j,X}$, by (3.6) and (3.7) we get

$$(\phi_X^* \circ \rho - \rho)_{0,0} = d_{0,1} \circ \Psi_X + \Psi_X \circ d_{0,1}. \tag{3.8}$$

Lemma 3.3. *$\Psi_X(\alpha)$ depends continuously on $X \in \overline{B^*}$ for each $\alpha \in A$ fixed.*

Proof. It is enough to prove that if $f: F \to \mathbb{R}$ is a continuous function then the function $\overline{f}: \overline{B^*} \to \mathbb{R}$ given by

$$\overline{f}(X) = \int_{F_X} f(X, Y, Z) \cdot d\mu_X(Y, Z)$$

is continuous. Take $X_0 \in \overline{B^*}$ and let $(X_n)_{n \in \mathbb{N}}$ be a sequence in $\overline{B^*}$ converging to X_0. Let

$$B_0 = B \cap \bigcap_{m=0}^{\infty} L_{a_m}^{-1} B,$$

where $a_m = \exp(X_m)$, being $\mu(\mathfrak{S} - B_0) = 0$. For $m = 0, 1, 2, \ldots$ take the continuous function $f_m: B_0 \to \mathbb{R}$ given by

$$f_m(g) = f(X_m, \log(g), \log(a_m \cdot g)).$$

$(f_n)_{n \in \mathbb{N}}$ converges pointwise to f_0 and for each $n \in \mathbb{N}$ we have $|f_n(g)| \leq \max\{|f(\xi)|/\xi \in F\}$ for all $g \in B_0$, therefore, by Lebesgue's Dominated Convergence Theorem and by Proposition 2.3 we obtain

$$\lim_{n \to \infty} \overline{f}(X_n) = \lim_{n \to \infty} \int_{B_0} f_n \cdot d\mu = \int_{B_0} f_0 \cdot d\mu = \overline{f}(X_0). \quad \square$$

Lemma 3.4. *For $\alpha \in A$, $X \in \mathfrak{g}$ and $t \in \mathbb{R}$ we have*

$$\phi_{tX}^* \alpha = \alpha + \int_0^t \phi_{sX}^* \circ \theta_X \alpha \cdot ds = \alpha + \theta_X \int_0^t \phi_{sX}^* \alpha \cdot ds.$$

Proof. For each $x \in M$ take the smooth curve $\gamma_x : \mathbb{R} \to \Lambda T_x^* M$ given by

$$\gamma_x(t) = \left(\phi_{tX}^* \alpha - \alpha - \int_0^t \phi_{sX}^* \alpha \cdot ds \right)(x).$$

We have $(d/dt)\gamma_k(t) = 0$ for all $t \in \mathbb{R}$, so $\gamma_x(t) = \gamma_x(0) = 0$. $\square$

Take $\alpha \in \operatorname{Ker}(d_{0,1})$ determining $[\alpha] \in E_1$ (by (1.2)). If $[\alpha] \in \rho_1(E_1)$ we can suppose $\alpha = \rho_{0,0}(\beta)$ for some $\beta \in \operatorname{Ker}(d_{0,1})$. Then, by (3.8) we have $(\phi_X^*)_{0,0}\alpha - \alpha = d_{0,1} \circ \Psi_X(\beta)$ for all $X \in \overline{B^*}$. Thus, by Lemma 3.3 and Lemma 3.4 it follows that

$$\begin{aligned}
(\theta_X)_{0,0}\alpha &= (\theta_X)_{0,0} \int_0^1 ((\phi_{sX}^*)_{0,0}\alpha - d_{0,1} \circ \Psi_{sX}(\beta)) \cdot ds \\
&= (\phi_X^*)_{0,0}\alpha - \alpha - d_{0,1} \circ (\theta_X)_{0,0} \int_0^1 \Psi_{sX}(\beta) \cdot ds \\
&= d_{0,1} \left(\Psi_X(\beta) - (\theta_X)_{0,0} \int_0^1 \Psi_{sX}(\beta) \cdot ds \right).
\end{aligned}$$

Therefore $\rho_1(E_1) \subset (E_1)_{\theta_1=0}$ and we can consider $\rho_1 \colon E_1 \to (E_1)_{\theta_1=0}$.

Now take $\alpha \in \operatorname{Ker}(d_{0,1})$ such that $[\alpha] \in (E_1)_{\theta_1=0}$. Then, since $(\theta_X)_{0,0}$ depends linearly on $X \in \mathfrak{g}$ there exists a linear map $X \mapsto \beta_X$ of $\mathfrak{g}$ to A so that $(\theta_X)_{0,0}\alpha = d_{0,1}(\beta_X)$ for all $X \in \mathfrak{g}$. Thus, by Lemma 3.4 we get

$$\rho_{0,0}(\alpha) = \alpha + d_{0,1} \int_{\overline{B^*}} \int_0^1 (\phi_{sX}^*)_{0,0}\beta_X \cdot ds \cdot \Delta^*(X), \tag{3.9}$$

obtaining $\rho_1([\alpha]) = [\alpha]$. Therefore, if $\imath \colon (E_1)_{\theta_1=0} \to E_1$ is the inclusion map we have $\rho_1 \circ \imath = \mathrm{id}$. We also have $\imath \circ \rho_1 - \mathrm{id} = d_1 \circ \lambda_1 + \lambda_1 \circ d_1$ obtaining

$$\imath_* = \rho_{1*}^{-1} \colon H((E_1)_{\theta_1=0}) \xrightarrow{\cong} E_2 . \tag{3.10}$$

Since $\mathfrak{G}$ is compact $\theta_{\mathfrak{g}}$ is a semisimple representation (§§4.4 and 5.12 in Volume III of [GHV]), so by (1.6) and by results of Volume III of [GHV] (Theorem V of §4.11 and §5.26) it follows that

$$H((E_1)_{\theta_1=0}) \cong E_2^{0,\cdot} \otimes H^{\cdot}(\mathfrak{g}) . \tag{3.11}$$

Therefore, from (3.10) and (3.11) we get

Theorem 3.5. *Let $\mathfrak{g}$ be a compact semisimple Lie algebra. For Lie $\mathfrak{g}$-foliations on compact manifolds we have $E_2^{u,v} \cong E_2^{0,v} \otimes H^u(\mathfrak{g})$ for u, v integers.*

Remark. One of the main geometrical properties related to the spectral sequence associated to an orientable and oriented foliation $\mathcal{F}$ on a connected manifold M is the minimality of the leaves for some Riemannian metric on M: *a smooth scalar product on $T\mathcal{F}$ is induced by a Riemannian metric on M for which all the leaves are minimal submanifolds if and only if the corresponding volume form on the leaves is the restriction to the leaves of a p-form which defines an element in $E_2^{0,p}$* [R, Su and Ha1]. This property is verified whenever $\mathcal{F}$ is an orientable Lie $\mathfrak{g}$-foliation with $\mathfrak{g}$ compact or nilpotent and M is compact [Ha1 and Ha2]. In particular it is verified under the hypothesis of the above theorem when the foliation is orientable, and in this case we obtain that if χ is the characteristic form associated to any Riemannian metric on M and any orientation of $\mathcal{F}$ [R] then the volume form along the leaves given by the restriction of $\rho(\chi)$ is induced by the orientation and the restriction to the leaves of a Riemannian metric on M for which all the leaves of $\mathcal{F}$ are minimal submanifolds (because $[\rho_{0,0}(\chi)] \in (E_1^{0,p})_{\theta_1=0} = E_2^{0,p}$).

Remark. The operators ρ_1 and λ_1 used to prove Theorem 3.5 are continuous, so they induce operators on $\mathscr{E}_1$ obtaining in a similar way (with a slight alteration of (3.9)) the analogous decomposition of $\mathscr{E}_2$ [A2]. Thus, by (1.4) we have $E_2 \cong \mathscr{E}_2$ canonically, and by (1.3) we get

Corollary 3.6. *With the same hypothesis, if the manifold is oriented we have $E_2^{u,v} \cong E_2^{q-u,v} \cong E_2^{u,p-v} \cong E_2^{q-u,p-v}$ for u, v integers.*

4. Riemannian foliations with compact semisimple structural Lie algebra

Theorem 4.1. *For Riemannian foliations on compact connected manifolds with compact semisimple structural Lie algebra we have $E_2 \cong \mathscr{E}_2$ canonically.*

Proof. The proof follows with similar arguments to those in §8 of [A2], they are shown here by the sake of completeness.

Let $(M, \mathscr{F})$ be a foliated manifold verifying the hypothesis and let $\mathfrak{g}$ be its structural Lie algebra. Firstly assume that $\mathscr{F}$ is transversely parallelizable and let $\pi_b: M \to W$ be the basic fibre bundle [Mo]. The fibres of π_b are the closures of the leaves of $\mathscr{F}$, and $\mathscr{F}$ induces on each fibre a Lie $\mathfrak{g}$-foliation.

We define the presheaves $\mathscr{P}_i$ and $\mathscr{Q}_i$ $(i = 1, 2)$ by setting

$$\mathscr{P}_i(U) = E_i(\mathscr{F}_{|\pi_b^{-1}(U)}) \quad \text{and} \quad \mathscr{Q}_i(U) = \mathscr{E}_i(\mathscr{F}_{|\pi_b^{-1}(U)})$$

for each open set U in W, and with the canonical restrictions. Then, fixed a good open covering $\mathscr{U}$ of W we have the Čech graded differential spaces $(\check{C}(\mathscr{U}, \mathscr{P}_i), \delta)$ and $(\check{C}(\mathscr{U}, \mathscr{Q}_i), \delta)$, and the differential operators $D = \delta + (-1)^k d_1$ on $\check{C}^k(\mathscr{U}, \mathscr{P}_1)$ and $D' = \delta + (-1)^k \mathscr{d}_1$ on $\check{C}^k(\mathscr{U}, \mathscr{Q}_1)$, where $\mathscr{d}_1$ is the differential operator on $\mathscr{E}_1$.

With a slight sharpening of the arguments of the Propositions 8.5 and 8.8 of [BT] we obtain that the restriction maps induce the isomorphisms $E_2(\mathscr{F}) \cong H(\check{C}(\mathscr{U}, \mathscr{P}_1), D)$ and $\mathscr{E}_2(\mathscr{F}) \cong H(\check{C}(\mathscr{U}, \mathscr{Q}_1), D')$. Thus we obtain spectral sequences converging to $E_2(\mathscr{F})$ and $\mathscr{E}_2(\mathscr{F})$ with second terms $H(\check{C}(\mathscr{U}, \mathscr{P}_2), \delta)$ and $H(\check{C}(\mathscr{U}, \mathscr{Q}_2), \delta)$ respectively (§14 of [BT]). Moreover, since $\mathscr{U}$ is a good open covering and the result is verified by Lie $\mathfrak{g}$-foliations on compact manifolds (second remark of Theorem 3.5) we obtain $H(\check{C}(\mathscr{U}, \mathscr{P}_2), \delta) \cong H(\check{C}(\mathscr{U}, \mathscr{Q}_2), \delta)$ canonically, so $E_2(\mathscr{F}) \cong \mathscr{E}_2(\mathscr{F})$.

In the general case, by standard arguments we can suppose that $\mathscr{F}$ is transversely oriented. Then let $\widehat{\mathscr{F}}$ be the horizontal lifting of $\mathscr{F}$ to the principal $SO(q)$-bundle of oriented orthonormal transverse frames with the transverse Levi-Civita connection [Mo]. The associated operation $(\mathfrak{o}(q), i, \theta, A(\widehat{M}), \hat{d})$ with the associated algebraic connection induces operations $(\mathfrak{o}(q), i_1, \theta_1, E_1(\widehat{\mathscr{F}}), \hat{d}_1)$ and $(\mathfrak{o}(q), \overline{i_1}, \overline{\theta_1}, \mathscr{E}_1(\widehat{\mathscr{F}}), \hat{d}_1)$ with the corresponding algebraic connections, being [A1 and A2]

$$E_2(\widehat{\mathscr{F}}) \cong H(E_1(\widehat{\mathscr{F}})_{\theta_1=0}), \quad E_2(\mathscr{F}) \cong H(E_1(\widehat{\mathscr{F}})_{\theta_1=0, i_1=0}),$$
$$\mathscr{E}_2(\widehat{\mathscr{F}}) \cong H(\mathscr{E}_1(\widehat{\mathscr{F}})_{\overline{\theta_1}=0}), \quad \mathscr{E}_2(\mathscr{F}) \cong H(\mathscr{E}_1(\widehat{\mathscr{F}})_{\overline{\theta_1}=0, \overline{i_1}=0}).$$

Thus, by results of Volume III of [GHV] (Corollary III of §9.5, Theorem III of §5.18 and Theorem I of §5.12) there exist spectral sequences converging to $E_2(\widehat{\mathscr{F}})$ and $\mathscr{E}_2(\widehat{\mathscr{F}})$ with second terms $E_2(\widehat{\mathscr{F}}) \otimes H^\bullet(\mathfrak{o}(q))$ and $\mathscr{E}(\widehat{\mathscr{F}}) \otimes H^\bullet(\mathfrak{o}(q))$ respectively. Therefore, since $\widehat{\mathscr{F}}$ is transversely parallelizable the result follows by §9.6 of Volume III of [GHV] and by Theorem 11.1 of Chapter XI of [Mc]. □

Corollary 4.2. *With the same hypothesis, if the manifold is oriented we have* $E_2^{u,v} \cong E_2^{q-u, p-v}$ *for* u, v *integers.*

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